View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 340 (2008) 53–63

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Proof of two conjectures of Móricz on double trigonometric series

Lee Tuo-Yeong

Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University,
1 Nanyang Walk, Singapore 637616, Republic of Singapore

Received 9 May 2007

Available online 14 August 2007

Submitted by Goong Chen

Abstract

We use a unified approach to obtain several integrability theorems of Boas [R.P. Boas, Integrability of trigonometric series, I, Duke Math. J. 18 (1951) 787–793]. In particular, we settle two conjectures of Móricz [F. Móricz, On the integrability of double cosine and sine series, II, J. Math. Anal. Appl. 154 (1991) 466–483] concerning double trigonometric series.
© 2007 Elsevier Inc. All rights reserved.

Keywords: Multiple sine series; Multiple cosine series; Regularly convergent multiple series

1. Introduction

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty} b_n = 0$ and $\sum_{k=1}^{\infty} |b_k - b_{k+1}|$ converges. It is well known that the sine series $\sum_{k=1}^{\infty} b_k \sin kx$ converges for all $x \in [0, \pi]$. Boas [1] proved that $\sum_{k=1}^{\infty} \frac{b_k}{k}$ converges if and only if $\lim_{\delta\to 0^+} \int_{\delta}^{\pi} \sum_{k=1}^{\infty} b_k \sin kx\, dx$ exists. Consequently, he proved that if $\sum_{k=0}^{\infty} a_k$ is an absolutely convergent series of real numbers and $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k = 0$, then $\lim_{\delta\to 0^+} \int_{\delta}^{\pi} \frac{1}{x}(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx)\, dx$ exists if and only if $\sum_{k=1}^{\infty} \frac{1}{k}(\frac{a_0}{2} + \sum_{j=1}^{k} a_j)$ converges.

The object of this paper is to use a unified method to extend the aforementioned results to higher dimensions; in particular, we give affirmative answers to two conjectures of Móricz [3] concerning double trigonometric series.

This paper is organized as follows. In Section 2 we deal with the one-dimensional case separately since it is straightforward and we get a good prospect of the general situation. In Section 3 we give some preliminaries on which all our subsequent reasoning depends. In Section 4 we obtain a crucial result concerning rectangular multiple series; see Theorem 4.3 for details. In Sections 5 and 6, we apply Theorem 4.3 to extend the above-mentioned Boas' results from one-dimensional to m-dimensional trigonometric series.

2. One-dimensional case

The main result of this section is Theorem 2.2 whose proof depends on the following well-known Dirichlet's test.

E-mail address: tuoyeong.lee@nie.edu.sg.

0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.08.010

Theorem 2.1. *Let $\{u_k\}_{k=1}^\infty$ and $\{v_k\}_{k=1}^\infty$ be two sequences of real numbers. If $\sup_{n\in\mathbb{N}}|\sum_{k=1}^n u_k|$ is finite, $\lim_{n\to\infty} v_n = 0$ and $\sum_{k=1}^\infty |v_k - v_{k+1}|$ converges, then $\sum_{k=1}^\infty u_k v_k$ converges and*

$$\sum_{k=1}^\infty u_k v_k = \sum_{k=1}^\infty (v_k - v_{k+1}) \sum_{j=1}^k u_j.$$

Theorem 2.2. *Let $\{c_k\}_{k=1}^\infty$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = 0$ and $\sum_{k=1}^\infty |c_k - c_{k+1}|$ converges. If $\{\Phi_n\}_{n=1}^\infty \subset C[0,\pi]$ and*

$$\sup_{n\in\mathbb{N}} \|\Phi_n\|_{C[0,\pi]} + \sup_{0<x<\pi}\sup_{n\in\mathbb{N}} \left|\frac{\Phi_n(x)-\Phi_n(0)}{nx}\right| + \sup_{x\in[0,\pi]}\sup_{n\in\mathbb{N}}\left|\sum_{k=1}^n x\Phi_k(x)\right| < \infty,$$

then $\sum_{k=1}^\infty \frac{c_k\Phi_k(x)}{k}$ converges for all $x \in (0,\pi]$. Moreover,

$$\lim_{\delta\to 0^+}\left\{\sum_{k=1}^{[\frac{1}{\delta}]}\frac{c_k\Phi_k(0)}{k} - \sum_{k=1}^\infty \frac{c_k\Phi_k(\delta)}{k}\right\} = 0. \tag{1}$$

Proof. The first assertion is a consequence of Dirichlet's test. To prove (1) we may assume that the sequence $\{c_k\}_{k=1}^\infty$ is decreasing and

$$\sup_{n\in\mathbb{N}} \|\Phi_n\|_{C[0,\pi]} + \sup_{0<x<\pi}\sup_{n\in\mathbb{N}} \left|\frac{\Phi_n(x)-\Phi_n(0)}{nx}\right| + \sup_{x\in[0,\pi]}\sup_{n\in\mathbb{N}}\left|\sum_{k=1}^n x\Phi_k(x)\right| \leqslant \frac{1}{2}.$$

Let $\delta \in (0,1)$ and let $[\frac{1}{\delta}]$ be the greatest integer less or equal to $\frac{1}{\delta}$. Then

$$\begin{aligned}\left|\sum_{k=1}^{[\frac{1}{\delta}]}\frac{c_k\Phi_k(0)}{k} - \sum_{k=1}^\infty \frac{c_k\Phi_k(\delta)}{k}\right| &\leqslant \sum_{k=1}^{[\frac{1}{\delta}]} c_k\left|\frac{\Phi_k(\delta)-\Phi_k(0)}{k}\right| + \left|\sum_{k=[\frac{1}{\delta}]+1}^\infty \frac{c_k\Phi_k(\delta)}{k}\right| \\ &\leqslant \frac{1}{[\frac{1}{\delta}]}\sum_{k=1}^{[\frac{1}{\delta}]} c_k + \frac{1}{[\frac{1}{\delta}]} c_{[\frac{1}{\delta}]} \sup_{n\geqslant[\frac{1}{\delta}]+1}\left|\sum_{k=[\frac{1}{\delta}]+1}^n \Phi_k(\delta)\right| \\ &\leqslant \frac{1}{[\frac{1}{\delta}]}\sum_{k=1}^{[\frac{1}{\delta}]} c_k + 2c_{[\frac{1}{\delta}]}.\end{aligned}$$

It is now clear that the assumption $\lim_{n\to\infty} c_n = 0$ implies (1). The proof is complete. □

The following results of Boas are immediate consequences of Theorem 2.2. Incidentally, we correct a small error (cf. [1, p. 792, lines 1–2]) in the proof of [1, Theorem 3(b)].

Theorem 2.3. *(See [1, Theorem 3(b)].) Let $\{b_k\}_{k=1}^\infty$ be a sequence of real numbers such that $\lim_{k\to\infty} b_k = 0$ and $\sum_{k=1}^\infty |b_k - b_{k+1}|$ converges. Then $\sum_{k=1}^\infty \frac{b_k}{k}$ converges if and only if $\lim_{\delta\to 0^+}\int_\delta^\pi \sum_{k=1}^\infty b_k \sin kx\,dx$ exists.*

Proof. We apply Theorem 2.2 with $c_k = b_k$ and $\Phi_k(x) = \int_x^\pi k\sin kt\,dt$ to conclude that $\sum_{k=1}^\infty b_k\frac{1-(-1)^k}{k}$ converges if and only if $\lim_{\delta\to 0^+}\sum_{k=1}^\infty \frac{b_k}{k}\int_\delta^\pi k\sin kt\,dt$ exists. Since it is well known that $\int_\delta^\pi \sum_{k=1}^\infty b_k\sin kx\,dx = \sum_{k=1}^\infty \int_\delta^\pi b_k \sin kx\,dx$ for all $0<\delta<\pi$ and the series $\sum_{k=1}^\infty (-1)^k\frac{b_k}{k}$ converges by Dirichlet's test, the theorem is proved. □

Theorem 2.4. *(See [1, Theorem 3(a)].) Let $\sum_{k=0}^\infty a_k$ be an absolutely convergent series of real numbers. If $\frac{a_0}{2} + \sum_{k=1}^\infty a_k = 0$, then $\lim_{\delta\to 0^+}\int_\delta^\pi \frac{1}{x}(\frac{a_0}{2} + \sum_{k=1}^\infty a_k\cos kx)\,dx$ exists if and only if $\sum_{k=1}^\infty \frac{1}{k}(\frac{a_0}{2} + \sum_{r=1}^k a_r)$ converges.*

Proof. Let $\lambda_0 := \frac{1}{2}$ and $\lambda_k := 1$ $(k \in \mathbb{N})$. If $x \in (0, \pi]$, then our assumptions and Theorem 2.1 yield

$$\begin{aligned}
\frac{1}{x}\sum_{k=0}^{\infty}\lambda_k a_k \cos kx &= -\frac{1}{x}\sum_{k=1}^{\infty} a_k(1-\cos kx)\\
&= -\frac{1}{x}\sum_{k=1}^{\infty}\left(\sum_{j=k}^{\infty} a_j\right)\bigl(\cos(k-1)x - \cos kx\bigr)\\
&= \frac{1}{x}\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k-1} \lambda_j a_j\right)\bigl(\cos(k-1)x - \cos kx\bigr)\\
&= \frac{a_0(1-\cos x)}{2x} + G(x)\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k}\lambda_j a_j\right)\sin\left(k+\frac{1}{2}\right)x,
\end{aligned}$$

where $G(x) := \frac{2\sin\frac{x}{2}}{x}$. Therefore

$$\lim_{\delta\to 0^+}\left\{\int_{\delta}^{\pi}\frac{1}{x}\sum_{k=0}^{\infty}\lambda_k a_k\cos kx\,dx - \int_{\delta}^{\pi} G(x)\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k}\lambda_j a_j\right)\sin\left(k+\frac{1}{2}\right)x\,dx\right\}\quad\text{exists.} \tag{2}$$

In view of (2) it remains to show that $\sum_{k=1}^{\infty}\frac{1}{k}(\frac{a_0}{2}+\sum_{r=1}^{k} a_r)$ converges if and only if

$$\lim_{\delta\to 0^+}\int_{\delta}^{\pi} G(x)\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k}\lambda_j a_j\right)\sin\left(k+\frac{1}{2}\right)x\,dx\quad\text{exists.}$$

Direct computations yield

$$\sup_{n\in\mathbb{N}}\sup_{x\in[0,\pi]}\left|\int_{x}^{\pi} nG(t)\sin\left(n+\frac{1}{2}\right)t\,dt\right| \leqslant \bigl(1+\|G'\|_{L^1[0,\pi]}\bigr)\sup_{n\in\mathbb{N}}\sup_{x\in[0,\pi]}\left|\frac{n\cos(n+\frac{1}{2})x}{n+\frac{1}{2}}\right| < \infty,$$

$$\sup_{x\in(0,\pi)}\sup_{n\in\mathbb{N}}\frac{1}{nx}\left|\int_{0}^{x} nG(t)\sin\left(n+\frac{1}{2}\right)t\,dt\right| < \infty,$$

and

$$\sup_{x\in[0,\pi]}\sup_{n\in\mathbb{N}}\left|x\sum_{k=1}^{n}\int_{x}^{\pi} kG(t)\sin\left(k+\frac{1}{2}\right)t\,dt\right| \leqslant 2\pi\|G'\|_{L^1[0,\pi]} + 2\pi^2 < \infty,$$

so we can apply Theorem 2.2 with $c_k = \sum_{j=1}^{k}\lambda_j a_j$ and $\Phi_k(x) = \int_x^{\pi} kG(t)\sin(k+\frac{1}{2})t\,dt$ to conclude that $\sum_{k=1}^{\infty}\frac{1}{k}(\sum_{j=0}^{k}\lambda_j a_j)\int_0^{\pi} kG(t)\sin(k+\frac{1}{2})t\,dt$ converges if and only if $\lim_{\delta\to 0^+}\int_{\delta}^{\pi}\frac{1}{x}(\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx)\,dx$ exists. Since $|\frac{1}{k}-\int_0^{\pi} G(x)\sin(k+\frac{1}{2})x\,dx| = O(\frac{1}{k^2})$ for all $k\in\mathbb{N}$ and the assumption $\sum_{k=0}^{\infty}|\lambda_k a_k| < \infty$ holds, the theorem follows. □

Theorem 2.5. *(See [1, Theorem 1(a)].) If $\sum_{k=1}^{\infty} b_k$ is an absolutely convergent series of real numbers, then $\lim_{\delta\to 0^+}\int_{\delta}^{\pi}\frac{1}{x}\sum_{k=1}^{\infty} b_k\sin kx\,dx$ exists.*

Proof. The proof is similar to that of Theorem 2.4. □

Theorem 2.6. *(See [1, Theorem 1(b)].) Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{k=0}^{\infty}|a_k - a_{k+1}|$ converges. Then $\lim_{\delta\to 0^+}\int_{\delta}^{\pi}(\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx)\,dx$ exists.*

Proof. The proof is similar to that of Theorem 2.3. □

3. Some preliminaries concerning multiple rectangular series

Unless specified otherwise, $m \geqslant 2$ is always a fixed positive integer. Each point $(x_1, \dots, x_m)$ in $\mathbb{R}^m$ is usually denoted by the corresponding bold letter $\boldsymbol{x}$. For $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}_0^m$, write $\boldsymbol{p} \leqslant \boldsymbol{q}$ or $\boldsymbol{q} \geqslant \boldsymbol{p}$ if and only if $p_i \leqslant q_i$ for each $i \in \{1, \dots, m\}$.

Definition 3.1. (See [2].) Let $\{u_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers. We consider the following (formal) multiple series:

$$\sum_{\boldsymbol{k} \in \mathbb{N}^m} u_{\boldsymbol{k}} := \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} u_{k_1,\dots,k_m}. \tag{3}$$

(i) For each $\boldsymbol{n} \in \mathbb{N}^m$, the rectangular $\boldsymbol{n}$ partial sum of (3) is given by

$$\sum_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} u_{\boldsymbol{k}} := \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m} u_{k_1,\dots,k_m}.$$

(ii) The multiple series (3) *converges in Pringsheim's sense* to a real number s if for each $\varepsilon > 0$ there exists an integer $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \sum_{\boldsymbol{1} \leqslant \boldsymbol{k} \leqslant \boldsymbol{n}} u_{\boldsymbol{k}} - s \right| < \varepsilon$$

whenever $\min\{n_1, \dots, n_m\} \geqslant N(\varepsilon)$.

(iii) The multiple series (3) *converges regularly* if for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \sum_{\boldsymbol{p} \leqslant \boldsymbol{k} \leqslant \boldsymbol{q}} u_{\boldsymbol{k}} \right| < \varepsilon$$

whenever $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^m$ with $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\max\{p_1, \dots, p_m\} \geqslant N(\varepsilon)$.

Theorem 3.2. *(See [2, Theorem 1].) Let $\{u_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers. The multiple series $\sum_{\boldsymbol{k} \in \mathbb{N}^m} u_{\boldsymbol{k}}$ is regularly convergent if and only if*

(i) *$\sum_{\boldsymbol{k} \in \mathbb{N}^m} u_{\boldsymbol{k}}$ converges in Pringsheim's sense, and*

(ii) *for each choice of the index $j \in \{1, \dots, m\}$ and for all fixed integral values of c_j, the $(m-1)$-multiple series*

$$\sum_{\substack{\boldsymbol{k} \in \mathbb{N}^m \\ k_j = c_j}} u_{\boldsymbol{k}}$$

are regularly convergent.

In order to formulate a multidimensional analogue of Theorem 2.1, we need some notations. Let $\{v_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers. For $j \in \{1, \dots, m\}$, set $\Delta_{\emptyset}(v_{\boldsymbol{k}}) := v_{\boldsymbol{k}}$ and

$$\Delta_j(v_{k_1,\dots,k_m}) := v_{k_1,\dots,k_{j-1},k_j,k_{j+1},\dots,k_m} - v_{k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_m}.$$

For $\{j_1, \dots, j_s\} \subseteq \{1, \dots, m\}$, set

$$\Delta_{\{j_1,\dots,j_s\}}(v_{\boldsymbol{k}}) := \Delta_{j_1}\bigl(\cdots\bigl(\Delta_{j_s}(v_{\boldsymbol{k}})\bigr)\cdots\bigr).$$

Set $|||\boldsymbol{x}||| = \max_{i=1,\dots,m} |x_i|$ $(\boldsymbol{x} \in \mathbb{R}^m)$. The following theorem is essential [2, Theorem 3].

Theorem 3.3. *Let $\{v_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequences of real numbers such that $\lim_{|||\boldsymbol{n}|||\to\infty} v_{\boldsymbol{n}} = 0$ and the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |\Delta_{\{1,\dots,m\}}(v_{\boldsymbol{k}})|$ converges. If the rectangular partial sums of the series* (3) *are bounded, then the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} u_{\boldsymbol{k}} v_{\boldsymbol{k}}$ converges regularly and*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} u_{\boldsymbol{k}} v_{\boldsymbol{k}} = \sum_{\boldsymbol{k}\in\mathbb{N}^m} \left\{ \Delta_{\{1,\dots,m\}}(v_{\boldsymbol{k}}) \sum_{\boldsymbol{1}\leqslant \boldsymbol{j}\leqslant \boldsymbol{k}} u_{\boldsymbol{j}} \right\}; \tag{4}$$

the multiple series on the right being absolutely convergent.

4. A multidimensional analogue of Theorem 2.2

The main result of this section is Theorem 4.3, which is an m-dimensional analogue of Theorem 2.2. We need the following lemmas.

Lemma 4.1. *Let $\{c_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers such that $\lim_{|||\boldsymbol{n}|||\to\infty} c_{\boldsymbol{n}} = 0$ and the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})|$ converges. Then there exist two multiple sequences $\{c_{1,\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$, $\{c_{2,\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ of non-negative numbers such that*

(a) $c_{\boldsymbol{k}} = c_{1,\boldsymbol{k}} - c_{2,\boldsymbol{k}}$ *for all* $\boldsymbol{k} \in \mathbb{N}^m$,
(b) $\lim_{|||\boldsymbol{n}|||\to\infty} c_{1,\boldsymbol{n}} = \lim_{|||\boldsymbol{n}|||\to\infty} c_{2,\boldsymbol{n}} = 0$, *and*
(c) *for* $i = 1, 2$, $0 \leqslant \Delta_{\{1,\dots,m\}}(c_{i,\boldsymbol{k}}) \leqslant |\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})|$ *for all* $\boldsymbol{k} \in \mathbb{N}^m$.

Proof. For each $i = 1, 2$ and $\boldsymbol{k} \in \mathbb{N}^m$, we set

$$c_{i,\boldsymbol{k}} := \frac{1}{2}\left\{ \sum_{\boldsymbol{r}\geqslant \boldsymbol{k}} \left(\left|\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{r}})\right| + (-1)^{i-1} \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{r}}) \right) \right\}.$$

Then it is clear that assertions (a)–(c) hold. □

Lemma 4.2. *If $u_1, \dots, u_m$ and $v_1, \dots, v_m$ are real numbers, then*

$$\left| \prod_{i=1}^m u_i - \prod_{i=1}^m v_i \right| \leqslant \sum_{j=1}^m \left(\prod_{i=1}^{j-1} |u_i| \right) \left(\prod_{i=j+1}^m |v_i| \right) |v_j - u_j|.$$

Proof. Use induction on m. □

Theorem 4.3. *Let $\{c_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers such that $\lim_{|||\boldsymbol{n}|||\to\infty} c_{\boldsymbol{n}} = 0$ and $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})| (\ln(|||\boldsymbol{k}||| + 1))^{m-1}$ converges. If $\{\Phi_{i,n}\}_{n=1}^\infty \subset C[0,\pi]$ $(i = 1, \dots, m)$ and*

$$\max_{i=1,\dots,m} \left\{ \sup_{n\in\mathbb{N}} \|\Phi_{i,n}\|_\infty + \sup_{x_i\in(0,\pi)} \sup_{n\in\mathbb{N}} \left| \frac{\Phi_{i,n}(x_i) - \Phi_{i,n}(0)}{n x_i} \right| + \sup_{x_i\in[0,\pi]} \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^n x_i \Phi_{i,k}(x_i) \right| \right\} < \infty,$$

then the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^m \frac{\Phi_{i,k_i}(x_i)}{k_i}$ converges regularly for all $\boldsymbol{x} \in (0,\pi]^m$ and

$$\lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0}\\ \boldsymbol{\delta}\in(0,\pi]^m}} \left\{ \sum_{\boldsymbol{1}\leqslant \boldsymbol{k}\leqslant ([\frac{1}{\delta_1}],\dots,[\frac{1}{\delta_m}])} c_{\boldsymbol{k}} \prod_{i=1}^m \frac{\Phi_{i,k_i}(0)}{k_i} - \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^m \frac{\Phi_{i,k_i}(\delta_i)}{k_i} \right\} = 0. \tag{5}$$

Moreover,

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^m \frac{\Phi_{i,k_i}(0)}{k_i} \quad \textit{converges regularly} \quad \Leftrightarrow \quad \lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0}\\ \boldsymbol{\delta}\in(0,\pi]^m}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^m \frac{\Phi_{i,k_i}(\delta_i)}{k_i} \quad \textit{exists.} \tag{6}$$

Proof. In view of Lemma 4.1 we may assume that

$$\lim_{|||\boldsymbol{n}|||\to\infty} c_{\boldsymbol{n}} = 0 \quad \text{and} \quad \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}}) \geqslant 0 \quad \bigl(\boldsymbol{k} \in \mathbb{N}^m\bigr). \tag{7}$$

Since $\min_{\boldsymbol{k}\in\mathbb{N}^m} \ln(|||\boldsymbol{k}||| + 1) \geqslant \ln 2 > \frac{1}{2}$, the first assertion is an easy consequence of Theorem 3.3. To prove (5) we may assume that

$$\max_{i=1,\dots,m} \left\{ \sup_{n\in\mathbb{N}} \|\Phi_{i,n}\|_\infty + \sup_{x_i\in(0,\pi)} \sup_{n\in\mathbb{N}} \left| \frac{\Phi_{i,n}(x_i) - \Phi_{i,n}(0)}{nx_i} \right| + \sup_{x_i\in[0,\pi]} \sup_{n\in\mathbb{N}} \left| \sum_{k=1}^{n} x_i \Phi_{i,k}(x_i) \right| \right\} \leqslant \frac{1}{2}.$$

Set $\Psi_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{i=1}^m \frac{\Phi_{i,k_i}(x_i)}{k_i}$ and let $\boldsymbol{\delta} \in (0,1)^m$ be given. We claim that

$$\begin{aligned} &\left| \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant([\frac{1}{\delta_1}],\dots,[\frac{1}{\delta_m}])} c_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\boldsymbol{\delta}) \right| \\ &\quad\leqslant \sum_{i=1}^m \frac{1}{[\frac{1}{\delta_i}]} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ 1\leqslant k_i\leqslant[\frac{1}{\delta_i}]}} \frac{k_i c_{\boldsymbol{k}}}{\prod_{j=1}^m k_j} + \sum_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} 12^m \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i\geqslant[\frac{1}{\delta_i}]\ \forall i\in\Gamma}} \bigl(\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})\bigr)\bigl(\ln\bigl(|||\boldsymbol{k}||| + 1\bigr)\bigr)^{m-1}. \end{aligned} \tag{8}$$

Set $\Gamma' := \{1,\dots,m\}\backslash\Gamma$ $(\Gamma \subseteq \{1,\dots,m\})$. Then, by the triangle inequality,

$$\begin{aligned} &\left| \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant([\frac{1}{\delta_1}],\dots,[\frac{1}{\delta_m}])} c_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\boldsymbol{0}) - \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\boldsymbol{\delta}) \right| \\ &\quad\leqslant \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant([\frac{1}{\delta_1}],\dots,[\frac{1}{\delta_m}])} c_{\boldsymbol{k}} \bigl|\Psi_{\boldsymbol{k}}(\boldsymbol{0}) - \Psi_{\boldsymbol{k}}(\boldsymbol{\delta})\bigr| + \sum_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} \left| \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i>[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ 1\leqslant k_i\leqslant[\frac{1}{\delta_i}]\ \forall i\in\Gamma'}} c_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\boldsymbol{\delta}) \right| \\ &\quad:= S + \sum_{\emptyset\neq\Gamma\subseteq\{1,\dots,m\}} |T_\Gamma|. \end{aligned} \tag{9}$$

A direct application of Lemma 4.2 yields

$$S \leqslant \sum_{i=1}^m \frac{1}{[\frac{1}{\delta_i}]} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ 1\leqslant k_i\leqslant[\frac{1}{\delta_i}]}} \frac{k_i c_{\boldsymbol{k}}}{\prod_{j=1}^m k_j}. \tag{10}$$

On the other hand, for each fixed non-empty $\Gamma \subseteq \{1,\dots,m\}$,

$$T_\Gamma = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ 1\leqslant k_i\leqslant[\frac{1}{\delta_i}]\ \forall i\in\Gamma'}} \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_i>k_i\ \forall i\in\Gamma \\ j_i=k_i\ \forall i\in\Gamma'}} c_{\boldsymbol{j}} \Psi_{\boldsymbol{j}}(\boldsymbol{\delta})$$

so that

$$|T_\Gamma| \leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ k_i\geqslant 1\ \forall i\in\Gamma'}} \left| \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_i>k_i\ \forall i\in\Gamma \\ j_i=k_i\ \forall i\in\Gamma'}} c_{\boldsymbol{j}} \prod_{i\in\Gamma} \frac{\Phi_{i,j_i}(\delta_i)}{j_i} \right| \prod_{i\in\Gamma'} \frac{1}{k_i}.$$

Using (7) and Theorem 3.3 we get

$$|T_\Gamma| \leqslant \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ k_i\geqslant 1\ \forall i\in\Gamma'}} \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^m \\ j_i=k_i\ \forall i\in\Gamma \\ j_i=k_i\ \forall i\in\Gamma'}} \frac{c_{\boldsymbol{j}}}{\prod_{i\in\Gamma} j_i} \prod_{i\in\Gamma} \frac{2}{\delta_i} \prod_{i\in\Gamma'} \frac{1}{k_i}. \tag{11}$$

Let R_Γ denote the right-hand side of (11). Then

$$\begin{aligned}
R_\Gamma &\leqslant 4^m \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ k_i\geqslant 1\ \forall i\in\Gamma'}} c_{\boldsymbol{k}} \prod_{i\in\Gamma'} \frac{1}{k_i} \\
&= 4^m \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ k_i\geqslant 1\ \forall i\in\Gamma'}} \left\{ \sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m \\ r_i\geqslant k_i\ \forall i\in\{1,\dots,m\}}} \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{r}}) \right\} \prod_{i\in\Gamma'} \frac{1}{k_i} \quad \big(\text{by (7)}\big) \\
&= 4^m \sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m \\ r_i\geqslant[\frac{1}{\delta_i}]\ \forall i\in\Gamma \\ r_i\geqslant 1\ \forall i\in\Gamma'}} \Delta_{\{1,\dots,m\}}(c_{\boldsymbol{r}}) \prod_{i\in\Gamma'} \sum_{k_i=1}^{r_i} \frac{1}{k_i} \quad \big(\text{using (7) and interchanging the order of summation}\big) \\
&\leqslant 12^m \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i\geqslant[\frac{1}{\delta_i}]\ \forall i\in\Gamma}} \big(\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})\big)\big(\ln\big(|||\boldsymbol{k}|||+1\big)\big)^{m-1}
\end{aligned} \tag{12}$$

because $\sum_{k=1}^{n} \frac{1}{k} \leqslant 3\ln(n+1)$ $(n \in \mathbb{N})$ and the cardinality of Γ is at most m. Combining (9)–(12) yields (8) to be proved.

Following the proof of (12) we get

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i=[\frac{1}{\delta_i}]}} \frac{k_i c_{\boldsymbol{k}}}{\prod_{j=1}^{m} k_j} \leqslant 3^{m-1} \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_i\geqslant[\frac{1}{\delta_i}]}} \big(\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})\big)\big(\ln\big(|||\boldsymbol{k}|||+1\big)\big)^{m-1} \tag{13}$$

for $i = 1, \dots, m$. Hence the assumption

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} \big|\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})\big|\big(\ln\big(|||\boldsymbol{k}|||+1\big)\big)^{m-1} < \infty$$

and (8) yield (5) to be proved.

It is now easy to obtain (6) as a consequence of Theorem 3.2. We infer from (5) that $\sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^{m} \frac{\Phi_{i,k_i}(0)}{k_i}$ converges in the sense of Pringsheim if and only if

$$\lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0} \\ \boldsymbol{\delta}\in(0,\pi]^m}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} c_{\boldsymbol{k}} \prod_{i=1}^{m} \frac{\Phi_{i,k_i}(\delta_i)}{k_i}$$

exists. From the proof of (13) it is clear that for $j \in \{1, \dots, m\}$ and for all fixed integral values of d_j, the $(m-1)$-multiple series

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m \\ k_j=d_j}} \frac{c_{\boldsymbol{k}}}{\prod_{i=1}^{m} k_i}$$

are regularly convergent. An appeal to Theorem 3.2 completes the proof. $\square$

5. An integrability theorem for multiple sine series

The main result of this section is Theorem 5.2, which gives an affirmative answer to a conjecture of Móricz [3, Remark 1(i)]. We need a lemma.

Lemma 5.1. *Let $\{b_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers such that $\lim_{|||\boldsymbol{n}|||\to\infty} b_{\boldsymbol{n}} = 0$ and $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |\Delta_{\{1,\ldots,m\}}(b_{\boldsymbol{k}})|(\ln(|||\boldsymbol{k}|||+1))^{m-1}$ converges. If $\Gamma \subseteq \{1,\ldots,m\}$ is non-empty, then*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i\in\Gamma} \frac{(-1)^{k_i-1}}{k_i} \prod_{i\in\Gamma'} \frac{1}{k_i}$$

is regularly convergent.

Proof. In view of Lemma 4.1, we may assume that $\Delta_{\{1,\ldots,m\}}(b_{\boldsymbol{k}}) \geqslant 0$ for all $\boldsymbol{k} \in \mathbb{N}^m$. If $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^m$ with $\boldsymbol{q} \geqslant \boldsymbol{p}$, then a single summation by parts gives

$$\begin{aligned}\left|\sum_{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}} b_{\boldsymbol{k}} \prod_{i\in\Gamma} \frac{(-1)^{k_i-1}}{k_i} \prod_{i\in\Gamma'} \frac{1}{k_i}\right| &\leqslant 2 \sum_{\substack{\boldsymbol{p}\leqslant\boldsymbol{k}\leqslant\boldsymbol{q}\\ k_\ell=p_\ell}} b_{\boldsymbol{k}} \prod_{\substack{i=1\\ i\neq\ell}}^{m} \frac{1}{k_i} \quad \text{for some } \ell\in\Gamma \\ &\leqslant 3^m \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^m\\ \boldsymbol{k}\geqslant\boldsymbol{p}}} \big(\Delta_{\{1,\ldots,m\}}(b_{\boldsymbol{k}})\big)\big(\ln\big(|||\boldsymbol{k}|||+1\big)\big)^{m-1}.\end{aligned}$$

It is now easy to check that the lemma holds. □

Theorem 5.2. *Let $\{b_{\boldsymbol{k}}\colon \boldsymbol{k} \in \mathbb{N}^m\}$ be a multiple sequence of real numbers such that $\lim_{|||\boldsymbol{n}|||\to\infty} b_{\boldsymbol{n}} = 0$ and $\sum_{\boldsymbol{k}\in\mathbb{N}^m} |\Delta_{\{1,\ldots,m\}}(b_{\boldsymbol{k}})|(\ln(|||\boldsymbol{k}|||+1))^{m-1}$ converges. Then the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i}$ converges regularly if and only if*

$$\lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0}\\ \boldsymbol{\delta}\in(0,\pi]^m}} \int_{\prod_{i=1}^m[\delta_i,\pi]} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \sin k_i x_i \, d\boldsymbol{x} \quad \textit{exists}. \tag{14}$$

Proof. We apply Theorem 4.3 with $c_{\boldsymbol{k}} = b_{\boldsymbol{k}}$ and $\Phi_{i,k}(x) = \int_x^\pi k \sin kt\, dt$ $(i = 1,\ldots,m)$ to conclude that (14) holds if and only if the multiple series

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \prod_{i=1}^m \int_{[0,\pi]} k_i \sin k_i x_i \, dx_i \quad \text{converges regularly.} \tag{15}$$

To this end, it remains to show that (15) holds if and only if $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i}$ is regularly convergent. But this assertion is a consequence of Lemma 5.1, since

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} \left\{ \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \prod_{i=1}^m \int_{[0,\pi]} k_i \sin k_i x_i \, dx_i - \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \right\}$$

can be written as a finite sum of regularly convergent multiple series

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} \left\{ \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \prod_{i=1}^m \int_{[0,\pi]} k_i \sin k_i x_i \, dx_i - \frac{b_{\boldsymbol{k}}}{\prod_{i=1}^m k_i} \right\} = \sum_{\emptyset\neq\Gamma\subseteq\{1,\ldots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i\in\Gamma} \frac{(-1)^{k_i-1}}{k_i} \prod_{i\in\Gamma'} \frac{1}{k_i}.$$

The proof is complete. □

6. An integrability theorem for multiple cosine series

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The main result of this section is Theorem 6.3, which gives an affirmative answer to another conjecture of Móricz [3, Remark 3(iii)]. We need two lemmas.

Lemma 6.1. *Let $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}}$ be an absolutely convergent multiple series of real numbers such that*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = 0 \quad \textit{for all } \boldsymbol{x} \in [0,\pi]^m \setminus (0,\pi]^m.$$

If $\boldsymbol{x} \in [0,\pi]^m$, then

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = (-1)^m \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}+\boldsymbol{1}} \prod_{i=1}^m \bigl(1-\cos(k_i+1)x_i\bigr). \tag{16}$$

Proof. For the case when $m=1$, it is easy to check that (16) holds. Since

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i} \cos k_i x_i = \sum_{k_1=0}^{\infty}\left\{\sum_{k_2=0}^{\infty}\left\{\cdots\left\{\sum_{k_m=0}^{\infty} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i}\cos k_i x_i\right\}\cdots\right\}\right\},$$

the lemma follows. □

Lemma 6.2. *Let $\{a_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}_0^m\}$ be given as in Lemma 6.1, let $\Gamma \subset \{1,\dots,m\}$, and let $\bigcup_{\ell\in\Gamma}\{c_\ell\} \subset \mathbb{N}_0$. Then*

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ k_\ell = c_\ell\ \forall \ell\in\Gamma \\ k_\ell \geqslant 0\ \forall\ell\in\Gamma'}} \left(\prod_{i\in\Gamma'} \lambda_{k_i}\right)(a_{\boldsymbol{k}}) = 0.$$

Proof. We may assume that Γ is non-empty. In this case, we have

$$\sum_{\substack{\boldsymbol{r}\in\mathbb{N}_0^m \\ r_\ell \geqslant 0\ \forall\ell\in\Gamma \\ r_\ell = 0\ \forall\ell\in\Gamma'}} \left\{\prod_{i\in\Gamma} \lambda_{r_i}\cos r_i t_i\right\}\left\{\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ k_\ell = r_\ell\ \forall\ell\in\Gamma \\ k_\ell \geqslant r_\ell\ \forall\ell\in\Gamma'}} \left(\prod_{i\in\Gamma'} \lambda_{k_i}\cos k_i(0)\right)(a_{\boldsymbol{k}})\right\} = 0$$

whenever $0 < t_i \leqslant \pi$ for each $i\in\Gamma$. Since

$$\sum_{\substack{\boldsymbol{r}\in\mathbb{N}_0^m \\ r_\ell \geqslant 0\ \forall\ell\in\Gamma \\ r_\ell = 0\ \forall\ell\in\Gamma'}} \left|\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^m \\ k_\ell = r_\ell\ \forall\ell\in\Gamma \\ k_\ell \geqslant r_\ell\ \forall\ell\in\Gamma'}} \left(\prod_{i\in\Gamma'} \lambda_{k_i}\cos k_i(0)\right)(a_{\boldsymbol{k}})\right| \leqslant \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} |a_{\boldsymbol{k}}| < \infty,$$

it is clear that the lemma holds. □

Theorem 6.3. *Let $\{a_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}_0^m\}$ be a multiple sequence of real numbers such that the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} |a_{\boldsymbol{k}}|(\ln(\|\boldsymbol{k}\|+2))^{m-1}$ converges and $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i}\cos k_i x_i = 0$ for all $\boldsymbol{x}\in[0,\pi]^m\setminus(0,\pi]^m$. Let*

$$f_1(\boldsymbol{x}) := \begin{cases} (\prod_{i=1}^m x_i)^{-1}(\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}} \prod_{i=1}^m \lambda_{k_i}\cos k_i x_i) & \textit{if } \boldsymbol{x}\in(0,\pi]^m, \\ 0 & \textit{otherwise.} \end{cases}$$

Then the multiple series $\sum_{\boldsymbol{k}\in\mathbb{N}^m} \prod_{i=1}^m \frac{1}{k_i} \sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} a_{\boldsymbol{j}} \prod_{i=1}^m \lambda_{j_i}$ converges regularly if and only if

$$\lim_{\substack{\boldsymbol{\delta}\to\boldsymbol{0} \\ \boldsymbol{\delta}\in(0,\pi]^m}} \int_{\prod_{i=1}^m[\delta_i,\pi]} f_1(\boldsymbol{x})\,d\boldsymbol{x} \quad \textit{exists.}$$

Proof. Let $\boldsymbol{x} \in (0,\pi]^m$ and set $A_{\boldsymbol{k}} := \sum_{\boldsymbol{0} \leqslant \boldsymbol{j} \leqslant \boldsymbol{k}} a_{\boldsymbol{j}} \prod_{i=1}^m \lambda_{j_i}$. Then

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= \frac{(-1)^m}{\prod_{i=1}^m x_i} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} a_{\boldsymbol{k}+\boldsymbol{1}} \prod_{i=1}^m \bigl(1 - \cos(k_i+1)x_i\bigr) \quad \text{(by Lemma 6.1)} \\
&= \frac{1}{\prod_{i=1}^m x_i} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} \Delta_{\{1,\dots,m\}}(A_{\boldsymbol{k}}) \prod_{i=1}^m \sum_{j_i=0}^{k_i} \bigl(\cos j_i x_i - \cos(j_i+1)x_i\bigr) \\
&= \frac{1}{\prod_{i=1}^m x_i} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} A_{\boldsymbol{k}} \prod_{i=1}^m \bigl(\cos k_i x_i - \cos(k_i+1)x_i\bigr) \\
&= \left\{ \prod_{i=1}^m \frac{2\sin\frac{x_i}{2}}{x_i} \right\} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} A_{\boldsymbol{k}} \prod_{i=1}^m \left(\sin k_i x_i \cos\frac{x_i}{2} + \cos k_i x_i \sin\frac{x_i}{2} \right) \\
&= \left\{ \prod_{i=1}^m \frac{2\sin\frac{x_i}{2}}{x_i} \right\} \sum_{\Gamma \subseteq \{1,\dots,m\}} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} A_{\boldsymbol{k}} \left\{ \prod_{i \in \Gamma} \sin k_i x_i \cos\frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos k_i x_i \sin\frac{x_i}{2} \right\} \\
&= \left\{ \prod_{i=1}^m \frac{2\sin\frac{x_i}{2}}{x_i} \right\} \sum_{\Gamma \subset \{1,\dots,m\}} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} A_{\boldsymbol{k}} \left\{ \prod_{i \in \Gamma} \sin k_i x_i \cos\frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos k_i x_i \sin\frac{x_i}{2} \right\} \\
&\quad + \left\{ \prod_{i=1}^m \frac{\sin x_i}{x_i} \right\} \sum_{\boldsymbol{k} \in \mathbb{N}^m} A_{\boldsymbol{k}} \prod_{i=1}^m \sin k_i x_i \\
&= g_1(\boldsymbol{x}) + g_2(\boldsymbol{x}) \quad \text{(say)}.
\end{aligned}
$$

We write $B_{\boldsymbol{k}} := \Delta_{\{1,\dots,m\}}(A_{\boldsymbol{k}})$. Then, by multiple summation by parts,

$$
\begin{aligned}
&\sup_{\Gamma \subset \{1,\dots,m\}} \int_{[0,\pi]^m} \left| \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} A_{\boldsymbol{k}} \left\{ \prod_{i \in \Gamma} \sin k_i x_i \cos\frac{x_i}{2} \right\} \left\{ \prod_{i \notin \Gamma} \cos k_i x_i \sin\frac{x_i}{2} \right\} \right| d\boldsymbol{x} \\
&\quad \leqslant \sup_{\Gamma \subset \{1,\dots,m\}} \int_{[0,\pi]^m} \sum_{\boldsymbol{k} \in \mathbb{N}_0^m} |B_{\boldsymbol{k}}| \left| \prod_{i \in \Gamma} \sum_{j_i=0}^{k_i} \sin j_i x_i \cos\frac{x_i}{2} \right| \left| \prod_{i \notin \Gamma} \sum_{j_i=0}^{k_i} \cos j_i x_i \sin\frac{x_i}{2} \right| d\boldsymbol{x} \\
&\quad = O\left(\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} |a_{\boldsymbol{k}}| \bigl(\ln\bigl(\|\!|\boldsymbol{k}|\!\| + 2\bigr)\bigr)^{m-1} \right) \\
&\quad < \infty,
\end{aligned}
$$

so $g_1 \in L^1([0,\pi]^m)$. To complete the proof, it remains to prove that the following statements are equivalent:

(a) $$\lim_{\substack{\boldsymbol{\delta} \to \boldsymbol{0} \\ \boldsymbol{\delta} \in (0,\pi]^m}} \int_{\prod_{i=1}^m [\delta_i,\pi]} g_2(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{exists.}$$

(b) $$\sum_{\boldsymbol{k} \in \mathbb{N}^m} A_{\boldsymbol{k}} \prod_{i=1}^m \int_0^\pi \frac{\sin x_i \sin k_i x_i}{x_i}\, dx_i \quad \text{converges regularly.}$$

(c) $$\sum_{\boldsymbol{k} \in \mathbb{N}^m} A_{\boldsymbol{k}} \prod_{i=1}^m \frac{1}{k_i} \quad \text{converges regularly.}$$

(a) $\Leftrightarrow$ (b). Direct computations yield

$$\sup_{n\in\mathbb{N}}\sup_{x\in[0,\pi]}\left|\int_x^\pi \frac{n\sin t\sin nt}{t}\,dt\right| \leqslant 2<\infty, \qquad \sup_{x\in(0,\pi)}\sup_{n\in\mathbb{N}}\frac{1}{nx}\left|\int_0^x \frac{n\sin t\sin nt}{t}\,dt\right|<\infty,$$

and

$$\sup_{x\in[0,\pi]}\sup_{n\in\mathbb{N}}\left|x\sum_{k=1}^{n}\int_x^\pi \frac{k\sin t\sin kt}{t}\,dt\right| \leqslant 2+2\pi<\infty,$$

so Theorem 4.3 implies that (a) is equivalent to (b).

Finally (b) is equivalent to (c) since

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in\mathbb{N}^m}|A_{\boldsymbol{k}}|\left|\prod_{i=1}^m\frac{1}{k_i}-\prod_{i=1}^m\int_0^\pi \frac{\sin x_i\sin k_i x_i}{x_i}\,dx_i\right| \\
&\quad\leqslant \sum_{\boldsymbol{k}\in\mathbb{N}^m}|A_{\boldsymbol{k}}|\left\{\sum_{j=1}^m\left(\prod_{i=1}^{j-1}\frac{1}{k_i}\right)\left|\prod_{i=j+1}^m\int_0^\pi \frac{\sin x_i\sin k_i x_i}{x_i}\,dx_i\right|\left|\frac{1}{k_j}-\int_0^\pi\frac{\sin x_j\sin k_j x_j}{x_j}\,dx_j\right|\right\} \quad \text{(by Lemma 4.2)} \\
&\quad=\sum_{j=1}^m\sum_{\boldsymbol{k}\in\mathbb{N}^m}O\left\{|A_{\boldsymbol{k}}|\left(\prod_{i=1}^{j-1}\frac{1}{k_i}\right)\left(\prod_{i=j+1}^m\frac{1}{k_i}\right)\left|\int_0^\pi\frac{\cos k_j x_j}{k_j}\frac{d}{dx_j}\left(\frac{\sin x_j}{x_j}\right)dx_j\right|\right\} \quad \text{(by integration by parts)} \\
&\quad=\sum_{j=1}^m\sum_{\boldsymbol{k}\in\mathbb{N}^m}O\left\{|A_{\boldsymbol{k}}|\left(\prod_{i=1}^{j-1}\frac{1}{k_i}\right)\left(\prod_{i=j+1}^m\frac{1}{k_i}\right)\frac{1}{k_j^2}\right\} \quad \text{(by integration by parts again)} \\
&\quad=\sum_{j=1}^m O\left(\sum_{\boldsymbol{k}\in\mathbb{N}^m}\left|\sum_{\boldsymbol{r}\geqslant\boldsymbol{k}}\Delta_{\{1,\dots,m\}}(A_{\boldsymbol{r}})\right|\left\{\left(\prod_{i=1}^{j-1}\frac{1}{k_i}\right)\left(\prod_{i=j+1}^m\frac{1}{k_i}\right)\frac{1}{k_j^2}\right\}\right) \\
&\quad=O\left(\sum_{\boldsymbol{k}\in\mathbb{N}^m}\left|\Delta_{\{1,\dots,m\}}(A_{\boldsymbol{k}})\right|\left(\ln\left(|||\boldsymbol{k}|||+1\right)\right)^{m-1}\right) \\
&\quad<\infty.
\end{aligned}$$

The proof is complete. □

References

[1] R.P. Boas, Integrability of trigonometric series, I, Duke Math. J. 18 (1951) 787–793.
[2] F. Móricz, Some remarks on the notion of regular convergence of multiple series, Acta Math. Hungar. 41 (1983) 161–168.
[3] F. Móricz, On the integrability of double cosine and sine series, II, J. Math. Anal. Appl. 154 (1991) 466–483.